Descent of Morphisms

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1 Introduction and Statement of the Main Theorem

Today we will prove that schemes are sheaves in the étale topology. However, with a little additional work we can show something more general, namely that schemes are sheaves in the fpqc topology. We start by defining the fpqc topology, which is sometimes done incorrectly in the literature, so we also include the incorrect definition.

Definition 1. *(WRONG!) A morphism of schemes $f : X \to Y$ is called quasicompact if Y can be covered by affine open subsets $\{V_i\}_{i \in I}$ such that $f^{-1}(V_i)$ is quasicompact, for all $i \in I$. A morphisms of schemes $\{U_i \to V\}_{i \in I}$ is called an fpqc cover is the map $\bigcup_{i \in I} \to Y$ is flat, surjective and quasicompact.*

The problem with this definition is that Zariski open immersions are not quasi-compact in general (c.f. [Sta16, Example 01K8]), however this is only a problem for non-Noetherian schemes. The correct definition is as follows:

Proposition 1 (Proposition 2.33 [Vis05])**.** *Let $f : X \to Y$ be a surjective morphism of schemes, then the following are equivalent*

1. *Every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.*
2. *There exists a covering $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.*
3. *Given a point $x \in X$, there exists an open neighborhood U of x in X, such that the image $f(U)$ is open in Y, and the restriction $U \to f(U)$ of f is quasi-compact.*
4. *Given a point $x \in X$, there exists a quasi-compact open neighborhood U of x in X, such that the image $f(U)$ is open and affine in Y.*

Definition 2. *A collection of morphisms $\{U_i \to V\}_{i \in I}$ is called an fpqc (fidèlement plate quasi-compacte) cover if the map $U = \bigcup_{i \in I} U_i \to V$ is flat, surjective and satisfies the equivalent conditions of Proposition 1.*

We are now in the position to state the main theorem that we will prove today:

Theorem 1 (Grothendieck)**.** *A representable functor on (Sch/S) is a sheaf in the fpqc topology.*

The proof consists of a some commutative algebra, which we will do at the end. At the heart of the proof lies a technical lemma, which is an interesting result in itself, we will proceed with this lemma in the next section.

Remark 1. *An fppf cover is an fpqc cover, so this theorem implies that schemes are sheaves in the fppf topology and hence in the étale topology*

2 A Technical Lemma

We start by proving the following Lemma

Lemma 1 ([Vis05], Lemma 2.60)**.** *Let S be a scheme, $F : \mathbf{Sch}/S \to \mathbf{Set}$ a contravariant functor. Suppose that F satisfies the following two conditions.*

(a) F is a sheaf in the Zariski Topology.

(b) Whenever $V \to U$ is faithfully flat morphism of affine S-schemes, the diagram

$$FU \longrightarrow FV \rightrightarrows F(V \times_U V)$$

is an equalizer.

Then F is a sheaf in the fpqc topology.

We will follow [Vis05] for the proof and follows his exposition and notation closely.

2.1 Reduction to the case of a single morphism

Let $\{U_i \to U\}_{i \in I}$ be an fpqc cover over S. We define $V = \coprod_{i \in I} U_i$ and let $F : V \to U$ be the map induced by the universal property of the coproduct. This gives us a new fpqc cover $\{V \to U\}$. We want to show that checking the sheaf property for the first cover is equivalent to checking the sheaf property for the second cover.

Note that the inclusions $U_i \to V$ are open immersions (essentially by definition of the disjoint union of schemes) hence we have a Zariski cover $\{U_i \to V\}_{i \in I}$. By hypothesis (a) we know that F is a Zariski sheaf, hence the following sequence is an equalizer.

$$F(V) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(j,k) \in I^2} F(U_j \times_V U_k).$$

However, the the fiber products $U_j \times_V U_k$ are just the intersection of U_j and U_k inside the disjoint union. In other words, there are empty unless $j = k$. This implies that we have a bijection

$$F(V) \leftrightarrow \prod_{i \in I} F(U_i).$$

Now, we can consider the diagram

$$\begin{array}{ccccc} F(U) & \longrightarrow & F(V) & \rightrightarrows & F(V \times_U V) \\ \Vert & & \downarrow & & \downarrow \\ F(U) & \longrightarrow & \prod_{i \in I} F(U_i) & \rightrightarrows & \prod_{(j,k) \in I^2} F(U_j \times_U U_k). \end{array} \tag{1}$$

Now, we use the fact that fiber products commute with disjoint unions to see that

$$\begin{aligned} V \times_U V &\cong \left(\coprod_i U_i \right) \times_U \left(\coprod_i U_i \right) \\ &\cong \coprod_{i,j} U_j \times_U U_k. \end{aligned}$$

By a similar argument as above, the maps $\{U_j \times_U U_k \to V \times_U V\}$ form a Zariski cover, hence

$$F(V \times_U V) \cong \prod_{(j,k) \in I^2} F(U_j \times_U U_k).$$

Now all columns of (1) are bijections, so the bottom row is an equalizer if and only if the top row is an equalizer. Therefore we can check the sheaf condition on coverings consisting of a single morphism. It is important to realize that both uniqueness of gluing and gluing can be checked on coverings consisting of a single morphism separately.

2.2 Seperatedness

Now let $f : V \to U$ be an fpqc morphism, we want to show that the map

$$F(U) \to F(V)$$

is injective. Take an open affine cover $\{U_i\}_{i \in I}$ of U and use fpqc-property to produce an open cover $\{V_i\}_{i \in I}$ of V with V_i quasi-compact and $F(V_I) = U_i$. Now choose a finite open affine cover $\{V_{i,\alpha}\}_{\alpha \in A_i}$ for each V_i . Now using these covers we can factor the map $F(U) \to F(V)$ as

$$\begin{array}{ccc} F(U) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ \prod_{i \in I} F(U_i) & \longrightarrow & \prod_{i \in I, \alpha \in A_i} F(V_{i,\alpha}). \end{array}$$

Once again, the columns are injective since F is a sheaf for the Zariski topology. We are left to show injectivity of the bottom map. This is equivalent to showing injectivity of

$$F(U_i) \to \prod_{\alpha \in A_i} F(V_{i,\alpha})$$

for all i. But, since F is a Zariski sheaf we know that

$$\prod_{\alpha \in A_i} F(V_{i,\alpha}) \cong F\left(\coprod_{\alpha \in A_i} V_{i,\alpha}\right)$$

and since A_i is finite for all i, we know that $\coprod_{\alpha \in A_i} V_{i,\alpha}$ is affine. By hypothesis (b) of the lemma, we get that

$$F(U_i) \to F(\coprod_{\alpha \in A_i} V_{i,\alpha})$$

is injective.

2.3 The case of a morphism from a quasi-compact scheme onto an affine scheme

Let $f : V \to U$ be a faithfully flat morphism with V quasi compact and U affine and let $b \in F(V)$ such that the arrows

$$F(V) \underset{\pi_1^*}{\overset{\pi_2^*}{\rightrightarrows}} F(V \times_U V)$$

agree. We want to find $a \in F(U)$ that maps to b under $F(f) : F(U) \to F(V)$. Since V is quasi compact we can take a finite affine open cover $V_1, \dots, V_n$ of V. The fact that $V \to U$ is an fpqc cover translates into the fact that $\coprod_{i=1}^n V_i \to U$ is an fpqc cover (since this is local on the target). Now $\coprod_{i=1}^n V_i$ is affine so using using hypothesis (b) of the Lemma, we get an equalizer diagram

$$F(U) \longrightarrow \prod_{i \in I} F(V_i) \underset{\pi_1^*}{\overset{\pi_2^*}{\rightrightarrows}} \prod_{i,j} V_i \times_U V_j.$$

Now both arrows still agree on $(b_1, \cdots, b_n)$ where $b_i := b\big|_{V_i}$ and so there is an element $a \in F(U)$ mapping to $(b_1, \cdots, b_n)$. Since F is a Zariski sheaf, the fact that $(Ff)(a)\big|_{V_i} = b_i$ for all i implies that $(Ff)(a) = b$.

2.4 The case of a morphism to an affine scheme

Let $f : V \to U$ be an fpqc morphism with U affine and let $b \in F(V)$ such that the arrows

$$F(V) \underset{\pi_1^*}{\overset{\pi_2^*}{\rightrightarrows}} F(V \times_U V)$$

agree. We want to find $a \in F(U)$ that maps to b under $F(f) : F(U) \to F(V)$.

Choose an open cover of V by quasi-compact open subschemes $\{V_i\}$ such that all maps $f\big|_{V_i} : V_i \to U$ are surjective (we claim that this is possible). Then by the fact that fpqc is source-local we see that $V_i \to U$ is an fpqc cover with V_i quasi-compact. Hence by the previous step we get (unique!) elements a_i in $F(U)$ that map to $b_i := b\big|_{V_i}$ for all i. However, we also get elements $a_{i,j}$ in $F(U)$ that map to $b_{i,j} := b\big|_{V_i \cup V_j}$ since $V_i \cup V_j \to U$ is also an fpqc cover.

Now, the map $V_i \to U$ factors through the inclusion $V_i \to V_i \cup V_i \to U$ hence the map

$$F(U) \to F(V_i)$$

factors through $F(V_i \cup V_j)$. Since the restriction of $b_{i,j}$ to U_i is b_i we get (by uniqueness of a_i!) that $a_i = a_{i,j} = a_j$.

Proof of claim: Let $x \in V$ and let V_x be an affine open neighborhood of x (which is automatically quasi compact). Next, let W be a quasi compact oepn subset of$V$ with image U, then $\{W \cup V_x\}_{x \in V}$ is an open cover of V by quasi-compact open subsets that surject onto U.

2.5 The general case

Let $f : V \to U$ be an arbitrary fpqc morphism, let $\{U_i\}_{i \in I}$ be an affine open cover of U and let $V_i = f^{-1}(U_i)$. By the previous step, the maps $V_i \to U_i$ induce equalizers

$$F(U_i) \longrightarrow F(V_i) \rightrightarrows F(V_i \times_{U_i} V_i),$$

which then induces a big equalizer

$$\prod_i F(U_i) \longrightarrow \prod_i F(V_i) \rightrightarrows \prod_{i,j} F(V_i \times_{U_i} V_i).$$

We can now put the sequence we are interested in as the top row of a commuting diagram

$$\begin{array}{ccccc}
F(U) & \longrightarrow & F(V) & \rightrightarrows & F(V \times_U V) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_i F(U_i) & \longrightarrow & \prod_i F(V_i) & \rightrightarrows & \prod_{i,j} F(V_i \times_{U_i} V_i) \\
\downarrow\downarrow & & \downarrow\downarrow & & \\
\prod_{i,j} F(U_i \times_U U_j) & \longrightarrow & \prod_{i,j} F(V_i \times_V V_j). & &
\end{array}$$

Once again, the left and middle columns are equalizers because F is a sheaf in the Zariski topology. The middle row is an equalizer by the discussion above. Since we already know that F is separated, we can do a diagram chase to show that the top row is an equalizer.

Informally, the following happens: Take $x \in F(V)$ such that the parallel arrows agree, then we get y in $\prod_i F(V_i)$ such that the parallel arrows agree, hence by the equalizer property we obtain $z \in \prod_i F(U_i)$ mapping to y.

Now, if we can show that the parallel arrows going down agree on z, we are done since the first column is an equalizer. But this is immediate from the fact that the bottom row is injective and the middle column an equalizer.

3 Proof of the Main Theorem

3.1 The Affine case

We start with a result from commutative algebra:

Lemma 2. *Let $f : A \to B$ be a faithfully flat morphisms of rings, i.e., the map* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is faithfully flat. Then the following sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{e_1 - e_2} B \otimes_A B$$

is exact.

Corollary 1. *This proves Theorem 1 in the case that X is affine. By Lemma 1 we only have to check the sheaf property for faithfully flap maps* $\operatorname{Spec} B \to \operatorname{Spec} A$. *Now write $X = \operatorname{Spec} R$, then we know that $h_X(-) \cong \mathbf{Sch}(-, X) \cong \mathbf{Sch}(-, \operatorname{Spec} R) \cong \mathbf{Ring}(R, -)$ and the functor $\mathbf{Ring}(R, -)$ is left exact. The Lemma now implies that the sequence*

$$0 \longrightarrow \mathbf{Ring}((R, A) \xrightarrow{f} \mathbf{Ring}((R, B) \xrightarrow{e_1 - e_2} \mathbf{Ring}((R, B \otimes_A B)$$

is exact, which is equivalent to saying that

$$\mathbf{Ring}(R, A) \longrightarrow \mathbf{Ring}((R, B) \underset{e_1}{\overset{e_2}{\rightrightarrows}} \mathbf{Ring}((R, B \otimes_A B)$$

is an equalizer diagram (exercise).

Proof of Lemma 2. Since the map $f : A \to B$ is faithfully flat, we know that it is injective. Indeed, we know for any ideal I of A that $I \otimes_A B = IB$ since B is flat. Now the fact that $0 = \ker(f)B = \ker f \otimes_A B$ implies (by faithfull flatness) that $\ker f = 0$.

Furthermore, it is clear that $(e_1 - e_2) \circ f = 0$ since the map f makes B into an A module. To be precise, we know that $1 \otimes f(a) = f(a) \otimes 1$ by the construction of $B \times_A B$ as an A-module. This implies that $\operatorname{Im} f \subset \ker(e_1 - e_2)$.

Last, we show that $\ker(e_1 - e_2) \subset \operatorname{Im} f$. Assume first that there is a section $g : B \to A$, i.e., a map such that $g \circ f : A \to A$ is the identity. Then take an element $b \in \ker(e_1 - e_2)$, then we can apply $g \times 1_B : B \times_A B \to A \times_A B = B$ and see

$$1 \otimes b = b \otimes 1$$
$$g(1) \otimes b = g(b) \otimes 1,$$

hence

$$f(1 \otimes b) = f(g(b) \otimes 1) = b,$$

so $b \in \operatorname{Im} f$.

Now, such a section might not exist, but we can check exactness after tensoring with B (since B is faithfully flat over A). In other words, it suffices to check exactness of the sequence

$$0 \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B.$$

We can fit this sequence into a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & B & \xrightarrow{f\otimes 1_B} & B\otimes_A B & \xrightarrow{e_1'-e_2'} & (B\otimes_A B)\otimes_B (B\otimes_A B) \\
 & & \Vert & & \Vert & & \downarrow{\scriptstyle \phi} \\
0 & \longrightarrow & B & \xrightarrow{f\otimes 1_B} & B\otimes_A B & \xrightarrow{(e_1-e_2)\otimes 1B} & (B\otimes_A B)\otimes_A B.
\end{array}$$

The top row is exact because there is a section (which is just the multiplication map). The bottom row is exact because the top row is exact and because ϕ is an isomorphism (follows from general nonsense). $\square$

3.2 The General Case

Let X be a scheme and choose an affine open cover $\{X_i\}_{i\in I}$ of X. By Lemma 1 we only have to check the sheaf condition for faithfully flat maps $h : V \to U$ of affine schemes U, V. We start by showing injectivity of the map

$$\mathbf{Sch}(U, X) \to \mathbf{Sch}(V, X).$$

Take two morphisms $f, g : U \to X$ such that the composites $V \to U \to X$ agree. Since h is surjective, we know that f, g agree as functions of sets. Now set $U_i = f^{-1}(X_i) = g^{-1}(X_i)$ and $V_i = h^{-1}(U_i)$. Since X_i is affine, we can apply Corollary 1 which shows that the following map is injective

$$\mathbf{Sch}(U_i, X_i) \to \mathbf{Sch}(V_i, X_i)$$

This means in particular that $f\big|_{U_i} = g\big|_{U_i}$ for all i, which implies that $f = g$.

Next, we show existence of gluing. Let $g \in \mathbf{Sch}(V, X)$ such that both arrows

$$\mathbf{Sch}(V, X) \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} \mathbf{Sch}(V \times_U V, X)$$

agree. In other words, we take a morphism

$$g : V \to X$$

such that the composites

$$V \times_U V \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} V \xrightarrow{g} X$$

agree. We want to show that there is a morphism $f \in \mathbf{Sch}(U, X)$ such that $g = f \circ h$, i.e., something that makes the diagram commute

$$\begin{array}{ccc}
V & \xrightarrow{h} & U \\
\downarrow{\scriptstyle g} & \swarrow{\scriptstyle f} & \\
X. & &
\end{array}$$

Define $V_i = g^{-1}(X_i)$ and $U_i = h(V_i)$. We know that V_i is open in V and since h is a quotient morphisms, it is open, so U_i is also open ([Sta16, Tag 02JY]). We know that the composites

$$V_i \times_{U_i} V_i \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} V_i \xrightarrow{g|V_i} X_i$$

agree. Since X_i is affine we know that the $g\big|_{V_i}$ factor uniquely through $f_i : U_i \to X_i$ (using that $V_i \to U_i$ is fpqc). We know that

$$f_i\big|_{U_{i,j}} = f_j\big|_{U_{i,j}}$$

since we can find a quasicompact open $V_{i,j}$ mapping fpqc to $U_{i,j}$ such that $g(V_{i,j} \subset X_i$. This implies that $g\big|_{V_{i,j}}$ factors uniquely through $f_{i,j} : U_{i,j} \to X_i$, hence $f_i\big|_{U_{i,j}} = fi, j$ and similarly with i, j reversed.

The above implies that the f_i glue together to a morphism $f : U \to X$ such that $g = f \circ h$. This shows that

$$\mathbf{Sch}(U, X) \longrightarrow \mathbf{Sch}(V, X) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{Sch}(V \times_U V, X)$$

is an equalizer diagram, which is what we wanted to show.

References

[Sta16] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2016.

[Vak] Ravi Vakil. *MATH 216: Foundations of Algebraic Geometry*. `http://math.stanford.edu/~vakil/216blog/`.

[Vis05] Angelo Vistoli. *Grothendieck topologies, fibered categories and descent theory*. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005. `http://homepage.sns.it/vistoli/descent.pdf`.

4 Exercises

1. *Show that any fppf covering is an fpqc covering (hint: use that fppf morphisms are open).*
2. *Show that $fpqc$ covers form a Grothendieck topology on* **Sch** *using Proposition 1.*
3. *Show that fiber products commute with disjoint unions in the category of schemes. To be precise, consider a family of schemes $\{U_i\}_{i\in I}$ with morphisms $f_i : U_i \to V$ and a scheme W with a morphisms $g : W \to V$. Show that*

$$\coprod_{i\in I}(U_i \times_V W) \cong \left(\coprod_{i\in I} U_i\right) \times_V W.$$

   (a) *Let $A, \{B_i\}_{i\in I}$ be locally ringed spaces (or schemes). Show that there is a natural bijection between*

$$\hom(A, \coprod_i B_i) \leftrightarrow \{\text{Partitions of } A \text{ into disjoint open sets } \{A_i\} \text{ and morphisms } A_i \to B_i\}.$$

   (b) *Use this, together with the universal property of the fiber product, to show that the functors*

$$\mathbf{Sch}\left(-, \coprod_{i\in I}(U_i \times_V W)\right)$$

   and

$$\mathbf{Sch}\left(-, \left(\coprod_{i\in I} U_i\right) \times_V W\right)$$

   are naturally isomorphic, then apply the Yoneda Lemma.

4. ([Vak] Exercise 9.4.D, [Vis05] Lemma 2.6.2) *Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be morphisms of schemes. Let x_1 and x_2 be points of X_1 and X_2 respectively such that $f_1(x_1) = f_2(x_2)$. Show that there exists a point z in the fibered product $X_1 \times_Y X_2$ such that $\pi_1(z) = x_1$ and $\pi_2(z) = x_2$. You may end up showing that for any fields k_1 and k_2 containing k_3, the ring $k_1 \otimes_{k_3} k_2$ is nonzero, and using the Axiom of Choice to find a maximal ideal in $k_1 \otimes_{k_3} k_2$.*
5. *Let R be a ring and let $f : A \to B$ and $g_1, g_2 : B \to C$ be morphisms of R-modules. Show that*

$$A \xrightarrow{f} B \overset{g_2}{\underset{g_1}{\rightrightarrows}} C$$

   is an equalizer diagram if and only if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g_1 - g_2} C$$

   is an exact sequence (both in the category of R-modules).